A NEW APPROACH TO PROVING THE GOLDBACH CONJECTURE

Anant Singh

Abstract:

The Goldbach's conjecture was first proposed in 1742 by Christian Goldbach in his letter to Leonard Euler, which goes as : "Every even integer greater than 2 can be written as the sum of two primes." It has fascinated us for the last 279 years as we can surely see it as being true, but can't give the proof for the same. In this paper we develop an easy approach to the problem using simple high school mathematics, including Euclid's Division Lemma and basic properties of even numbers, which may eventually lead us to a new way to prove it.

1. Introduction:

The Goldbach's Conjecture remains to this day one of the oldest unproven conjectures in the history of mathematics. In its modern form, it states that every even number larger than two can be expressed as a sum of two prime numbers. Its statement is very simple but yet proving it has proven to be really hard for us. Although it has been shown that the conjecture holds for all integers less than 4×10^{18} [1], but we are still lacking a rigorous proof for it. The object of this paper is to give a rather different approach to this problem, and come up with an easier method to show a new way to prove it, and pave the way for a fully complete proof. We will show all the different ways of representing even numbers and prime numbers. Then, we will separately present the method to express a particular form of even numbers as the sum of two prime numbers, and will do this for all the forms of even numbers.

2. Representation of prime and even numbers:

We will show the method to represent the prime and even numbers in different ways using the Euclid's Division Lemma.

Lemma 2.1: If we have two positive integers a and b, then there would be whole numbers q and r that satisfy the equation

$$a = bq + r \quad ; \; 0 \leq r \leq b \quad (2.1)$$

Where a is the dividend, b is the divisor, q is the quotient and r is the remainder.

Lemma 2.2. Every positive even number is of the form 2n, where n is any positive integer. Now, we are going to show the different ways to represent prime and even numbers. So, from equation 2.1 and taking $b = 6$, we have ,

$$a = 6q + r\,;\ 0 \leq r \leq 6$$

Thus, the possible values of *r* are as follows :

$$r = 0, 1, 2, 3, 4, 5$$

Now, we are going to put these values of *r* one by one in Equation 2.2

1. For *r* = 0,

$$a = 6q + 0$$
$$a = 6q$$
$$a = 2(3q)$$

   Thus, a is even. And since it is even, it can't be prime (except for 2)

2. For *r* = 1,

$$a = 6q + 1$$

   Since the equation is in its simplest form and we cannot factorize it further, so we conclude that *a* is prime.

3. For *r* = 2,

$$a = 6q + 2$$
$$a = 2(3q + 1)$$

   Thus, a is even. And since it is even, it cannot be prime (except for 2, which indeed is the answer if we put $q = 0$. And because of this, we will take 2 as a prime separately later on.)

4. For *r* = 3,

$$a = 6q + 3$$
$$a = 3(2q + 1)$$

   Thus, *a* is a multiple of 3 and so, it cannot be prime. (Although 3 is itself a prime, and it is indeed the answer of the equation for $q = 0$. And because of this, we will take 3 as a prime separately later on.)

5. For *r* = 4,

$$a = 6q + 4$$
$$a = 2(3q + 2)$$

   Thus, *a* is even. And since it is even, it cannot be prime (except for 2)

6. For $r = 5$,

$$a = 6q + 5$$

Since the equation is in its simplest form and we cannot factorize it further, so we conclude that a is prime.

Thus, we conclude the following:

- Every even number is of the form $6q$ or $6q + 2$ or $6q + 4$.
- Every prime number is of the form $6q + 1$ or $6q + 5$ (except for the primes 2 and 3).

3. Some basic results:

Now, we are going to prove some more basic results which will help us to approach our final proof. We have,

$$6q + 5$$

Taking $q = s - 1$,

$$6(s - 1) + 5$$

$$6s - 6 + 5$$

$$6s - 1$$

Thus, we can write $6q + 5$ as $6s - 1$. We will now use this proof and from now on write $6q + 5$ as $6s - 1$ in all future references. Also, we have,

$$6q + 4$$

Taking $q = s - 1$,

$$6(s - 1) + 4$$

$$6s - 6 + 4$$

$$6s - 2$$

Thus, we can write $6q + 4$ as $6s - 2$ or vice-versa. This will be used later in our approach to the proof.

4. Approach to the proof:

Since we have now developed the prerequisites to proceed further, we can now move on to our next step. We have seen from section 2 that every prime number (except 2 and 3) is of the form $6q + 1$ or $6q + 5$. Also, from section 3, we can now write this as follows:

Every prime number is of the form $6q + 1$ *or* $6s - 1$ *(except 2 and 3)*

Thus every possible prime number is one of the following:

- 2
- 3
- $6n + 1$
- $6m - 1$

The conjecture is based on the sum of two prime numbers, so we will now work out all the possible sum of two prime numbers. The results are as follows:

i. $2 + 2 = 4$
ii. $2 + 3 = 5$
iii. $2 + (6q + 1) = 6q + 3$
iv. $2 + (6s - 1) = 6s + 1$
v. $3 + 3 = 6$
vi. $3 + (6q + 1) = 6q + 4$
vii. $3 + (6s - 1) = 6s + 2$
viii. $(6q + 1) + (6s - 1) = 6q + 6s = 6(q + s)$
ix. $(6q + 1) + (6s + 1) = 6q + 6s + 2 = 6(q + s) + 2$
x. $(6q - 1) + (6s - 1) = 6q + 6s - 2 = 6(q + s) + 4$ (from 3)

The primes for cases (ix) and (x) are chosen so as to account for the fact that they may not be the successor or predecessor of the same multiple of 6.

Out of all these, the ones which are in the form of even numbers (as we've proved in 2) are (i), (v), (vi), (vii), (viii), (ix) and (x). Clearly, (i) and (v) give us the exact solution for the even numbers 4 and 6 respectively. Now, we have a statement which is equivalent to the Goldbach conjecture:

For a given even number 'a', we can always find at least one pair of prime numbers whose sum is equal to 'a'. And, we can do this for each and every even number.

Thus, we may now prove it for the remaining even numbers, i.e., even numbers greater than 6. Now, we are going to show the method to find the two primes whose sum is the given even number. We will show it case by case for every way of expressing the even numbers.

Let *a* be an even number:

1) For $a = 6q$:
   The only sum which corresponds to this type of even numbers is (viii). Thus, this even number can be expressed as a sum of two primes, which are of the form $6q + 1$ and $6s - 1$ respectively. For the computation of these two primes, we can proceed as follows :

$$a = 6(q + s)$$

$$\frac{a}{6} = q + s$$

And thus, we can find all the possible values of q and s and then substitute them in $6q + 1$ and $6s - 1$ respectively to check if they yield a pair of primes or not. If not, we check the next value pair and repeat the process until we find a prime pair.

2) For $a = 6q + 2$:
There are two sums which corresponds to even numbers of the form $6q + 2$. These are (vii) and (ix). If the case is (vii), we check if $(a - 3)$ is a prime number or not. If yes, then the two primes whose sum is a are 3 and $(a - 3)$. If not, then it is case (ix) and now we will proceed as,

$$a = 6(q + s) + 2$$
$$a - 2 = 6(q + s)$$
$$\frac{a-2}{6} = q + s$$

And then we proceed as done earlier, find all the possible values of q and s and then substitute them in $6q + 1$ and $6s + 1$ respectively to check if they yield a pair of primes or not. If not, we check the next value pair and repeat the process until we find a prime pair.

3) For $a = 6q + 4$:
There are two sums which corresponds to even numbers of the form $6q + 4$. These are (vi) and (x). If the case is (vi), we check if $(a - 3)$ is a prime number or not. If yes, then the two primes whose sum is a are 3 and $(a - 3)$. If not, then it is case (x) and now we will proceed as,

$$a = 6(q + s) + 4$$
$$a - 4 = 6(q + s)$$
$$\frac{a-4}{6} = q + s$$

And then again, we find all the possible values of q and s and then substitute them in $6q - 1$ and $6s - 1$ respectively to check if they yield a pair of primes or not. If not, we check the next value pair and repeat the process until we find a prime pair.

Example: We now show an example for the above mentioned method. Suppose the given even number is 38. So, $a = 38$. Since we know it leaves a remainder of 2 when divided by 6, thus it is an even number of the type $6q + 2$. Thus, we first check if $(a - 3)$ is prime.

$$a - 3 = 38 - 3$$
$$a - 3 = 35$$

and we know that 35 is not prime. Now, we will proceed as :

$$38 = 6(q + s) + 2$$
$$38 - 2 = 6(q + s)$$

$$36 = 6(q + s)$$
$$\frac{36}{6} = q + s$$
$$6 = q + s$$

So, we now check for the value-pairs, which we will get as : (1,5), (2,4), (3,3), (4,2), (5,1). Now, we check these value-pairs to get our prime pair(s).

- For (1,5):
$$6q + 1 = 6(1) + 1 = 7$$
$$6s + 1 = 6(5) + 1 = 31$$
Which are both primes. Thus, we get a prime pair (7, 31)

- For (2,4):
$$6q + 1 = 6(2) + 1 = 13$$
$$6s + 1 = 6(4) + 1 = 25$$
Since 25 is not a prime, thus, it’s not a prime pair.

- For (3,3):
$$6q + 1 = 6(3) + 1 = 19$$
$$6s + 1 = 6(3) + 1 = 19$$
Which are both primes. Thus, we get a prime pair (19, 19)

- For (4,2):
$$6q + 1 = 6(4) + 1 = 25$$
$$6s + 1 = 6(2) + 1 = 13$$
Since 25 is not a prime, thus, it’s not a prime pair

- For (5,1):
$$6q + 1 = 6(5) + 1 = 31$$
$$6s + 1 = 6(1) + 1 = 7$$
Which are both primes. Thus, we get a prime pair (31, 7)

So, we got 2 distinct prime pairs which are (31, 7) and (19, 19). And we can easily verify:

$$31 + 7 = 38$$

$$19 + 19 = 38$$

5. Conclusion:

We have shown the method to compute the two primes whose sum is the given even number on case by case basis, and showed that this can be done for any type of even number. And this proves statement 4, which is just a restatement of the Goldbach conjecture. And thus, we arrive at our proof for the Goldbach Conjecture. But, as we can notice, this proof hinges on the fact that:

For every even number, among all the possible value-pairs of q+s, there always exists at least one pair of values that results in a pair of primes.

Since, for every even number we can find two prime numbers whose sum is the given even number, and have shown that it holds for every even number less than 4×10^{18}, I am assuming the above statement 5 to be true. The proof is completed by taking statement 4 as a hypothesis, it is strongly desired if we can somehow prove the statement 4, which will finally prove the Goldbach's Conjecture rigorously.

References:

1. Tom`as Oliveira e Silva, Siegfried Herzog, and Silvio Pardi, Empirical verification of the even goldbach conjecture and computation of prime gaps up to 4×10^{18} ., Math. Comput. 83 (2014), no. 288, 2033–2060.

Current address: K-4/10B, Mohan Garden, Uttam Nagar, New Delhi, India - 110059

Email address: sanant711@gmail.com